Sequences and Series 1

1. The first three of four integers are in an A.P. and the last three are in G.P.. Find these four numbers, given that the sum of the first and the last integers is 37 and the sum of the two integers in the middle is 36.

Let the numbers be a, b, c and d.

$a + c = 2b \quad \dots(1)$

$c^2 = bd \quad \dots(2)$

$a + d = 37 \quad \dots(3)$

$b + c = 36 \quad \dots(4)$

From (3), $d = 37 - a \quad \dots(5)$

Substitute (5) in (2), $c^2 = b(37 - a) \quad \dots(6)$

From (1), $a = 2b - c \quad \dots(7)$

Substitute (7) in (6), $c^2 = b(37 - 2b + c) \quad \dots(8)$

From (4), $b = 36 - c \quad \dots(9)$

Substitute (9) in (8), $c^2 = (36 - c)[37 - 2(36 - c) + c]$

$$4\,c^2 - 143\,c + 1260 = 0$$

$$\therefore c = \frac{63}{4} \text{ or } c = 20$$

Since c is an integer, $c = 20$.

From (9), $b = 16$

From (7), $a = 12$

From (5), $d = 25$

The four numbers are $12, 16, 20, 25$.

2. The 3rd, 6th and 12th terms of an A.P, are successive terms of a G.P. Show that 4th, 8th and 16th terms of the A.P. are also successive terms of a G.P.

$T(3) = a + 2d, T(6) = a + 5d, T(12) = a + 11d$

Since they are in G.P., $(a + 5d)^2 = (a + 2d)(a + 11d)$

$a^2 + 10ad + 25d^2 = a^2 + 13ad + 22d^2$

$3d^2 = 3d$

Since $d \neq 0$, $a = d$. (If d = 0, the result is trivial.)

$(a + 7d)^2 = (a + 7a)^2 = 64a^2$

$(a + 3d)(a + 15d) = (a + 3a)(a + 15a) = 64a^2$

Therefore $(a + 7d)^2 = (a + 3d)(a + 15d)$.

$$T(8)^2 = T(4)T(16)$$

Therefore, 4th, 8th and 16th terms of the A.P. are also successive terms of a G.P.

3. Show that the sum of the odd numbers from 1 to (2n – 1) inclusive is n^2. Show that the sum of positive odd numbers smaller than 1002 that cannot be divided by 3 is 6×167^2.

$S = 1 \qquad + 3 \qquad + 5 + \ \ldots \qquad + (2n-1)$
$S = (2n-1) + (2n-3) + (2n-5) + \cdots + 3 + 1$
Adding, we get $2S = n(2n)$.
$\therefore S = n^2$
$S = 1 + 3 + 5 + \cdots + 1001 = 501^2 = (3 \times 167)^2 = 9 \times 167^2$
$S_1 = 3 + 9 + \cdots + 999 = 3(1 + 3 + \cdots + 333) = 3 \times 167^2$
The sum of positive odd numbers smaller than 1002 that cannot be divided by 3
$= S - S_1 = 9 \times 167^2 - 3 \times 167^2 = 6 \times 167^2$.

4. The sum, $\mathrm{S_n}$ of the first n terms of the sequence $\mathrm{u_1, u_2, u_3, \ldots}$ is $\mathrm{S_n = n(3n - a)}$, where a is a constant.

(a) Find $\mathrm{u_n}$ in terms of a and n.
(b) Find the recurrence relation of $\mathrm{u_n}$ in the form of $\mathrm{u_{n+1} = f(u_n)}$.

(a) $\mathrm{u_n = S_n - S_{n-1} = n(3n-a) - (n-1)[3(n-1) - a] = 3n^2 - an - 3(n^2 - 2n + 1) + an - a}$
$= \mathbf{6n - (a+3)}$
(b) $\mathrm{u_{n+1} - u_n = [6(n+1) - (a+3)] - [6n - (a+3)] = 6}$
$\therefore \mathbf{u_{n+1} = u_n + 6}$

5. A sequence $\mathrm{u_1, u_2, u_3, \ldots}$ is such that $\mathrm{u_1 = 1}$ and $\mathrm{u_{n+1} = 4u_n + 7}$ for $n \geq 1$.
Write down the first four terms of the sequence,

Show that an explicit formula for $\mathrm{u_r}$ is given by $\mathrm{u_r = 1 + \frac{10}{3}[4^{r-1} - 1]}$

$\mathrm{u_1 = 1,\ u_2 = 11,\ u_3 = 51,\ u_4 = 211}$.
$\mathrm{u_n - u_{n-1} = (4u_{n-1} + 7) - (4u_{n-2} + 7) = 4(u_{n-1} - u_{n-2}) \ldots (1)}$
Lower the index of (1) by 1, we get $\mathrm{u_{n-1} - u_{n-2} = 4(u_{n-2} - u_{n-3})}$
Hence, $\mathrm{u_n - u_{n-1} = 4(u_{n-1} - u_{n-2}) = 4[4(u_{n-2} - u_{n-3})] = 4^2(u_{n-2} - u_{n-3})}$
$\mathrm{= \cdots = 4^{n-2}(u_2 - u_1) = 4^{n-2}(10)}$
Hence, $\mathrm{u_r - u_{r-1} = 4^{r-2}(10)}$
$\mathrm{u_{r-1} - u_{r-2} = 4^{r-3}(10)}$
……
$\mathrm{u_2 - u_1 = 4^0(10)}$
Adding, $\mathrm{u_r - u_1 = 10[4^{r-2} + 4^{r-3} + \cdots + 1]}$, which is a geometric series

$\mathrm{= 10\frac{4^{r-1} - 1}{4 - 1}}$

$\mathrm{\therefore u_r = u_1 + 10\frac{4^{r-1} - 1}{4 - 1} = 1 + \frac{10}{3}[4^{r-1} - 1]}$

6. Given $u_n = e^n - 1$. Prove that the sequence is partly a geometric progression. Hence find the value of $\sum_{r=1}^{n} u_r$.

$v_r = e^r,\ v_{r-1} = e^{r-1}$, $\frac{v_r}{v_{r-1}} = \frac{e^r}{e^{r-1}} = e$, a constant.

Therefore v_r is a geometric.

$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n}(e^r - 1) = \sum_{r=1}^{n} e^r - \sum_{r=1}^{n} 1 = \boldsymbol{e\left(\frac{e^n-1}{e-1}\right) - n}$

7. **(a)** Show that for a fixed number $x \neq 1$, $3x^2 + 3x^3 + \cdots + 3x^n$ is a geometric series and find its sum in terms of x and n.

(b) The series $T_n(x) = x + 4x^2 + 7x^3 + \cdots + (3n-2)x^n$, for $x \neq 1$.
By considering $T_n(x) - xT_n(x)$ and using the result from (a), show that

$$T_n(x) = \frac{x+2x^2-(3n+1)x^{n+1}+(3n-2)x^{n+2}}{(1-x)^2}.$$

Hence, find the value of $\sum_{r=1}^{20} 2^r\,(3r-2)$ and deduce the value of $\sum_{r=0}^{19} 2^{r+2}\,(3r+1)$

(a) Let $u_r = 3x^{r+1},\ u_{r-1} = 3x^r,\ \frac{u_r}{u_{r-1}} = \frac{3x^{r+1}}{3x^r} = 3$, which is a constant.

Therefore, $3x^2 + 3x^3 + \cdots + 3x^n$ is a geometric series.

$3x^2 + 3x^3 + \cdots + 3x^n = 3x^2\left(\frac{1-x^{n-1}}{1-x}\right) = 3\left(\frac{x^2-x^{n+1}}{1-x}\right),\ x \neq 1$

(b) $T_n(x) = x + 4x^2 + 7x^3 + \cdots + (3n-2)x^n$
$xT_n(x) = \quad +x^2 + 4x^3 + \cdots + (3n-5)x^n + (3n-2)x^{n+1}$
$T_n(x) - xT_n(x) = x + (3x^2 + 3x^3 + \cdots + 3x^n) - (3n-2)x^{n+1}$

$(1-x)T_n(x) = x + 3\left(\frac{x^2-x^{n+1}}{1-x}\right) - (3n-2)x^{n+1}$, by (a).

$$= \frac{x(1-x)+3(x^2-x^{n+1})+(3n-2)x^{n+1}(1-x)}{(1-x)} = \frac{x+2x^2-(3n+1)x^{n+1}+(3n-2)x^{n+2}}{(1-x)}$$

$\therefore T_n(x) = \frac{x+2x^2-(3n+1)x^{n+1}+(3n-2)x^{n+2}}{(1-x)^2}$

$T_n(x) = \sum_{r=1}^{20} x^r\,(3r-2) = \frac{x+2x^2-(3n+1)x^{n+1}+(3n-2)x^{n+2}}{(1-x)^2}$

Put n = 20, x = 2, we have $\sum_{r=1}^{20} 2^r\,(3r-2) = \frac{2+2(2^2)-(3(20)+1)2^{20+1}+(3(20)-2)(2^{20+2})}{(1-2)^2}$

$= \mathbf{115343370}$

$\sum_{r=0}^{19} 2^{r+2}\,(3r+1) = \sum_{i=1}^{20} 2^{(i-1)+2}\,(3(i-1)+1)$ (replace the index r by i =r + 1)
$= \sum_{i=1}^{20} 2^{i+1}\,(3i-2) = 2\sum_{i=1}^{20} 2^i\,(3i-2) = 2 \times 115343370 = \mathbf{230686740}$

8. **(a)** Use partial fractions to show that:

$$\frac{2}{1\times3\times5}+\frac{3}{3\times5\times7}+\frac{4}{5\times7\times9}+\frac{5}{7\times9\times11}\ldots+\frac{n+1}{(2n-1)(2n+1)(2n+3)}=\frac{n(5n+7)}{6(2n+1)(2n+3)}$$

(b) State whether the series $\sum_{r=1}^{n}\frac{r+1}{(2r-1)(2r+1)(2r+3)}$ converges as $n\to\infty$ and if it does, find its sum to infinity.

(a) Consider the general term

$$\frac{r+1}{(2r-1)(2r+1)(2r+3)}\equiv\frac{A}{2r-1}+\frac{B}{2r+1}+\frac{C}{2r+3}$$

$$A(2r+1)(2r+3)+B(2r-1)(2r+3)+C(2r-1)(2r+1)\equiv r+1$$

Put $r=\frac{1}{2},\ 8A=\frac{3}{2}, A=\frac{3}{16}$

Put $r=-\frac{1}{2}, -4B=\frac{1}{2},\ B=-\mathbf{\frac{1}{8}}$

Put $r=-\frac{3}{2},\ 8C=-\frac{1}{2},\ C=-\frac{1}{16}$

$$\frac{r+1}{(2r-1)(2r+1)(2r+3)}\equiv\frac{\frac{3}{16}}{2r-1}-\frac{\frac{1}{8}}{2r+1}-\frac{\frac{1}{16}}{2r+3}=\frac{1}{8}\left[\frac{1}{2r-1}-\frac{1}{2r+1}\right]+\frac{1}{16}\left[\frac{1}{2r-1}-\frac{1}{2r+3}\right]$$

$$\sum_{r=1}^{n}\frac{r+1}{(2r-1)(2r+1)(2r+3)}=\frac{1}{8}\sum_{r=1}^{n}\left[\frac{1}{2r-1}-\frac{1}{2r+1}\right]+\frac{1}{16}\sum_{r=1}^{n}\left[\frac{1}{2r-1}-\frac{1}{2r+3}\right]$$

$$=\frac{1}{8}\left[\frac{1}{2(1)-1}-\frac{1}{2n+1}\right]+\frac{1}{16}\left[\frac{1}{2(1)-1}+\frac{1}{2(2)-1}-\frac{1}{2(n-1)+3}-\frac{1}{2n+3}\right]$$

$$=\frac{5}{24}-\frac{3}{16}\left(\frac{1}{2n+1}\right)-\frac{1}{16}\left(\frac{1}{2n+3}\right)=\frac{10(2n+1)(2n+3)-9(2n+3)-3(2n+1)}{48}=\frac{8\,n\,(5\,n+7)}{48}=\frac{n(5n+7)}{6(2n+1)(2n+3)}$$

(b) $\sum_{r=1}^{\infty}\frac{r+1}{(2r-1)(2r+1)(2r+3)}=\lim_{n\to\infty}\frac{n(5n+7)}{6(2n+1)(2n+3)}=\lim_{n\to\infty}\frac{\left(5+\frac{7}{n}\right)}{6\left(2+\frac{1}{n}\right)\left(1+\frac{3}{n}\right)}=\mathbf{\frac{5}{12}}$

Method 2

Let $u_r=\frac{r+1}{(2r-1)(2r+1)(2r+3)}$, $v_r=\frac{r(5r+7)}{6(2r+1)(2r+3)}$ (Hehe! compare with RHS of what to prove)

$$v_r-v_{r-1}=\frac{r(5r+7)}{6(2r+1)(2r+3)}-\frac{(r-1)(5r+2)}{6(2r-1)(2r+1)}=\frac{r(5r+7)(2r-1)-(r-1)(5r+2)(2r+3)}{6(2r-1)(2r+1)(2r+3)}=\frac{6(r+1)}{6(2r-1)(2r+1)(2r+3)}$$

Hence $\sum_{r=1}^{\infty}\frac{r+1}{(2r-1)(2r+1)(2r+3)}=\sum_{r=1}^{\infty}u_r=\sum_{r=1}^{\infty}(v_r-v_{r-1})=v_n-v_0=\frac{n(5n+7)}{6(2n+1)(2n+3)}$

9. Express $\frac{1}{(3r-2)(3r+1)}$ in partial fractions.

Show that $\sum_{r=1}^{n}\frac{1}{(3r-2)(3r+1)}=\frac{1}{3}\left[1-\frac{1}{3n+1}\right]$.

Hence , find $\sum_{r=1}^{\infty}\frac{1}{(3r-2)(3r+1)}$.

$\frac{1}{(3r-2)(3r+1)} \equiv \frac{A}{3r-2} + \frac{B}{3r+1} \Longrightarrow \mathrm{A}(3r+1) + B(3r-2) \equiv 1$

Put $r = \frac{2}{3}$, $A = \frac{1}{3}$ and put $r = -\frac{1}{3}$, $B = -\frac{1}{3}$

$\therefore \frac{1}{(3r-2)(3r+1)} \equiv \frac{1}{3}\left(\frac{1}{3r-2} - \frac{1}{3r+1}\right)$

$\sum_{r=1}^{n} \frac{1}{(3r-2)(3r+1)} = \frac{1}{3}\left(1 - \frac{1}{4}\right) + \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7}\right) + \frac{1}{3}\left(\frac{1}{7} - \frac{1}{10}\right) + \cdots + \frac{1}{3}\left(\frac{1}{3n-2} - \frac{1}{3n+1}\right) = \frac{1}{3}\left[1 - \frac{1}{3n+1}\right]$

$\sum_{r=1}^{\infty} \frac{1}{(3r-2)(3r+1)} = \lim_{n\to\infty} \frac{1}{3}\left[1 - \frac{1}{3n+1}\right] = \boldsymbol{\frac{1}{3}}$

10. Express $u_r = \frac{2}{(r+1)(r+3)}$ in partial fractions.

Using the result obtained,

(i) show that $u_r^{\,2} = -\frac{1}{r+1} + \frac{1}{r+3} + \frac{1}{(r+1)^2} + \frac{1}{(r+3)^2}$,

(ii) show that $\sum_{r=1}^{n} u_r = \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}$, and determine the values of

$\sum_{r=1}^{\infty} u_r$ and $\sum_{r=1}^{\infty} \left(u_{r+1} + \frac{1}{2^r}\right)$.

$u_r = \frac{2}{(r+1)(r+3)} \equiv \frac{1}{r+1} - \frac{1}{r+3}$

(i) $u_r^{\,2} = \left(\frac{1}{r+1} - \frac{1}{r+3}\right)^2 = \frac{1}{(r+1)^2} - \frac{2}{(r+1)(r+3)} + \frac{1}{(r+3)^2} = \frac{1}{(r+1)^2} - \left[\frac{1}{r+1} - \frac{1}{r+3}\right] + \frac{1}{(r+3)^2}$

$= -\frac{1}{r+1} + \frac{1}{r+3} + \frac{1}{(r+1)^2} + \frac{1}{(r+3)^2}$

(ii) $\sum_{r=1}^{n} u_r = \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right) + \left(\frac{1}{n+1} - \frac{1}{n+3}\right)$

$= \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} = \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}$

$\sum_{r=1}^{\infty} u_r = \lim_{n\to\infty} \left[\frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}\right] = \boldsymbol{\frac{5}{6}}$

$\sum_{r=1}^{\infty} \left(u_{r+1} + \frac{1}{2^r}\right) = \sum_{r=1}^{\infty} u_{r+1} + \sum_{r=1}^{\infty} \frac{1}{2^r} = (u_2 + u_3 + u_4 + \cdots) + \sum_{r=1}^{\infty} \frac{1}{2^r} = \sum_{r=1}^{\infty} u_r - u_1 + \sum_{r=1}^{\infty} \frac{1}{2^r}$

$= \frac{5}{6} - \frac{2}{(1+1)(1+3)} + \frac{\frac{1}{2}}{1-\frac{1}{2}} = \boldsymbol{\frac{19}{12}}$ (The second sum is an infinite G.P., use $S(\infty) = \frac{a}{1-r}$.)

Yue Kwok Choy

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